Algebraically explicit analytical solutions of two-buoyancy natural convection in porous media*

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Abstract Analytical solutions of governing equations of various physical phenomena have their own irreplaceable theoretical meaning. In addition, they can also be the benchmark solutions to verify the outcomes and codes of numerical solution, and to develop various numerical methods such as their differencing schemes and grid generation skills as well. In order to promote the development of the discipline of natural convection, three simple algebraically explicit analytical solution sets are derived for a non-linear simultaneous partial differential equation set with five dependent unknown variables, which represents the natural convection in porous media with both temperature and concentration gradients. An extraordinary method separating variables with addition is applied in this paper to deduce solutions.

Keywords: explicit analytical solution, natural convection, porous medium, temperature gradient, concentration gradient.

Heat and mass transfer in porous media is an important process in many fields such as in petroleum and geothermal reservoir engineering, thermal technique in building, nuclear reactor design, environment protection, casting technology, drying technology and so forth. It is necessary to research such phenomena in depth.

The two-buoyancy natural convection in porous media with both temperature and concentration gradients is studied in this paper. Such phenomenon has not yet been researched in detail. The steady geometrically 2-D governing equation sets of such a phenomenon is given by Ref. [1] as follows:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u=-\frac{\partial p}{\partial x}, \tag{2}$$

$$v=-\frac{\partial p}{\partial y}-R_u+R_a\theta-NR_aC, \tag{3}$$

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}, \tag{4}$$

$$\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}=L_e\left(u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}\right), \tag{5}$$

where u and v are the velocity components in x and y coordinates; p is pressure; θ and C are non-dimensional temperature and concentration; N, Ra, Le and Ru are ratio of buoyancy forces, Rayleigh number, Lewis number and non-dimensional parameter representing gravity respectively. A detailed numerical solution of Eqs. (1) ~ (5) with all non-dimensional parameters being equal to constant is given by Ref. [1], and some qualitative rules are given also.

Analytical solutions of governing equations of various disciplines have been very important for their development. For example, the analytical solutions of incompressible flow and constant coefficient heat conduction in early days have been the bases of fluid dynamics and heat transfer[2,3]. However, to derive analytical solutions with given initial and boundary conditions are very difficult, especially for the complex governing equations. In addition, owing to the rapid development of computers and numerical methods, concrete problems tend to be solved with computational fluid dynamics and computational heat and mass transfer. Nevertheless, analytical solutions have their own irreplaceable theoretical meaning. They can also be the benchmark solutions to develop numerical methods, for example, to verify the accuracy, convergence and effectiveness of various numerical approaches, and to improve various numerical methods such as their differencing schemes and grid generation skills as well. For example, several analytical solutions which can simulate the 3-D potential flow in turbomachine cascades were given by the first author[4] about twenty years ago, which have been used

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successfully by some references of computational fluid dynamics. Therefore, some possible algebraically explicit analytical solutions of abovementioned Eqs. (1 ~ 5) are derived in this paper to develop the heat and mass transfer related to the two-buoyancy natural convection in porous media with both temperature and concentration gradients.

Since the main aim is to find possible analytical solutions but not solutions for given initial and boundary conditions, the derivation approach in this paper is different from the common method. We find the possible solutions of Eqs. (1 ~ 5) first, and then decide what their boundary conditions are. Such an approach is similar to the derivation of typical basic analytical solutions of incompressible fluid dynamics in early time.

The abovementioned basic equation set is in strict sense a non-linear simultaneous partial differential equation set with five dependent unknown variables. Even all coefficients are constants, it is not easy to derive solutions, so some extraordinary approaches are applied in the following paragraphs.

1 The first algebraically explicit analytical solution set

For deriving analytical solutions of partial differential equations, we have applied a simple extraordinary method[5, 6]—a method of separating variables with addition. In this approach, we assumed an unknown function $f(t, x) = T(t) + X(x)$ instead of $f(t, x) = T(t)X(x)$ in the common method. It is found that this approach can obtain solutions of some equations having not yet obtained any analytical solutions before. In this paragraph, we apply this extraordinary approach to the abovementioned non-linear simultaneous basic equation set with five dependent unknown variables to obtain analytical solutions.

Actually, the basic equation (1) can be satisfied with the two-dimensional stream function ψ commonly used in fluid dynamics. For incompressible flow, $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$. Then the basic equation set can be degenerated into a simultaneous equation set with only four dependent variables.

If we assume that

$$\psi = X_1(x) + Y_1(y), \quad p = X_2(x) + Y_2(y),$$

$$\theta = X_3(x) + Y_3(y), \quad C = X_4(x) + Y_4(y),$$

substituting abovementioned four expressions into governing equations (2 ~ 4), we obtain:

$$Y_1' = -X_2', \tag{6}$$

$$-X_1' = -Y_2' - Ru + Ra(X_3 + Y_3) - NRa(X_4 + Y_4), \tag{7}$$

$$X_3'' + Y_3'' = Y_1' X_3' - X_1' Y_3', \tag{8}$$

$$X_4'' + Y_4'' = Le(Y_1' X_4' - X_1' Y_4'). \tag{9}$$

From Eq. (6), it is deduced:

$$Y_1 = K_1 y + K_2, \tag{10}$$

$$X_2 = K_3 - K_1 x, \tag{11}$$

where K_1, K_2 and K_3 are arbitrary constants. In following equations, K_i represent different arbitrary constants.

Substituting Eq. (10) into Eqs. (8 ~ 9), it is obvious that X_1' has to be constant in order to make separating variables possible (it is able to separate variables with $X_3'' = 0 = Y_4''$ also, but there is a difficulty later to derive solutions, we give up such simplification). With $X_1' = -K_4$, then

$$v = -X_1' = K_4. \tag{12}$$

Eqs. (8) and (9) then can be separated as:

$$Y_3'' - K_4 Y_3' = K_6 = -X_3'' + K_1 X_3', \tag{13}$$

$$Y_4'' - K_4 Le Y_4' = K_{10} = -X_4'' + K_1 Le X_4'. \tag{14}$$

Solving both sides of Eqs. (13 ~ 14), it is obtained:

$$Y_3 = \frac{K_7}{K_4} e^{K_4 y} - \frac{K_6}{K_4} y + K_5, \tag{15}$$

$$Y_4 = \frac{K_{11}}{K_4 Le} e^{K_4 Le y} - \frac{K_{10}}{K_4 Le} y + K_8, \tag{16}$$

$$X_3 = \frac{K_9}{K_1} e^{K_1 x} + \frac{K_6}{K_1} x, \tag{17}$$

$$X_4 = \frac{K_{12}}{K_1 Le} e^{K_1 Le x} + \frac{K_{10}}{K_1 Le} x. \tag{18}$$

Then, the only function that has to be solved now is Y_2. Substituting Eqs. (12) and (15 ~ 18) into Eq. (7), it is understood that there are two possibilities to separate variables to derive Y_2. The first one needs $K_9 = 0 = K_{12}$, $K_6 = NK_{10}/Le$; the second one needs $Le = 1$, $K_9 = K_{12}N$ and $K_6 = K_{10}N$. Substituting the first condition into Eq. (7), it is derived:

$$Y_2 = -(K_4 + Ru - K_5 Ra + NRaK_8)y + Ra \frac{K_7}{K_4^2} e^{K_4 y} - NRa \frac{K_{11}}{(K_4 Le)^2} e^{K_4 Le y}. \tag{19}$$

Combining Eqs. (10 ~ 12) and Eqs. (15 ~ 19), it is able to obtain an algebraically explicit analytical solution set of governing equations (1 ~ 5):

$$\psi = K_1 y + K_2 - K_4 x \, (u = K_1, \, v = K_4), \tag{20}$$

$$p = K_3 - K_1 x - (K_4 + Ru - K_5 Ra + K_8 NRa) y + Ra \frac{K_7}{K_4^2} e^{K_4 y} - NRa \frac{K_{11}}{(K_4 Le)^2} e^{K_4 Ley}, \tag{21}$$

$$\theta = \frac{K_6}{K_1} x + \frac{K_7}{K_4} e^{K_4 y} - \frac{K_6}{K_4} y + K_5, \tag{22a}$$

$$C = \frac{K_6}{K_1 N} x + \frac{K_{11}}{K_4 Le} e^{K_4 Ley} - \frac{K_6 y}{K_4 N} + K_8 \tag{23a}$$

Substituting the second condition mentioned above, with similar derivation, it is deduced that Eqs. (20 ~ 21) are still effective, but Eqs. (22a) and (23a) have to be replaced by following equations:

$$\theta = \frac{K_{12} N}{K_1} e^{K_1 x} + \frac{K_6}{K_1} x + \frac{K_7}{K_4} e^{K_4 y} - \frac{K_6}{K_4} y + K_5, \tag{22b}$$

$$C = \frac{K_{12}}{K_1} e^{K_1 x} + \frac{K_6}{K_1 N} x + \frac{K_{11}}{K_4} e^{K_4 y} - \frac{K_6}{K_4 N} y + K_8. \tag{23b}$$

This solution needs $Le = 1$ and its applicable range is more narrow than the previous solution, but it can still be a benchmark solution for computational fluid dynamics and computational heat and mass transfer. Moreover, if $N = 1$, $K_7 = K_{11}$ and $K_5 = K_8$, the non-dimensional temperature field and non-dimensional concentration field of this solution are exactly the same, which is a special solution. The abovementioned two solutions are similar, but they are independent of each other. The former can include the influence of Le, while the latter only describes the special condition of $Le = 1$.

The velocities of both solutions are constant, then it is unable to simulate the natural convection in a not inclined and rectangular box as done by Chen et al.[1] But at least they can strictly and simply reflect the physical condition of some element in a large porous material. The boundary condition of the element just satisfies what is described by Eqs. (20) ~ (23). They are still useful as benchmark solutions.

If $K_1 = 0 = K_6 = K_{12}$, both solutions describe the natural convection in porous media between two infinite vertical parallel solid plates moving up or down with the same velocity. In this case, the fluid in the porous media is moving with the same velocity as the plates, and there are no temperature and concentration gradients across the solid plate boundary; the pressure, temperature and concentration are functions of y only. Even the physical boundary conditions are clear, but actually these two are one-dimensional solutions. Of course, they can be the benchmark solutions also.

Furthermore, when all constants $K_i \neq 0$, these solutions describe the natural convection between two infinite inclined solid plates with the gradient equal to K_4 / K_1. However, there will be temperature and concentration gradients across the solid plate boundary.

2 The second algebraically explicit analytical solution set

Similar to the previous paragraph, deducing Eqs. (6) ~ (12) with the approach of separating variables with addition, and then assuming $K_4 = 0$, it is possible to obtain the following four relations instead of Eqs. (15) ~ (18):

$$Y_3 = K_6 y^2 / 2 + K_7 y + K_8, \tag{24}$$

$$Y_4 = K_{10} y^2 / 2 + K_{11} y + K_{13}, \tag{25}$$

$$X_3 = \frac{K_9}{K_1} e^{K_1 x} + \frac{K_6}{K_1} x, \tag{26}$$

$$X_4 = \frac{K_{12}}{K_1 Le} e^{K_1 Lex} + \frac{K_{10}}{K_1 Le} x, \tag{27}$$

Substituting Eqs. (24) ~ (27) into Eq. (7), similar to the previously described, it is understood that the Y_2 can be derived only when $K_9 = 0 = K_{12}$ and $K_6 = NK_{10} / Le$, or $Le = 1$, $K_9 = K_{12} N$ and $K_6 = K_{10} N$. For the first case, the total solution is as follows:

$$\psi = K_1 y + K_2 \quad (u = K_1, \, v = 0), \tag{28}$$

$$p = K_3 - K_1 x - [Ru - Ra(K_8 - NK_{13})] y + Ra \frac{K_7 - K_{11} N}{2} y^2 + \frac{K_6}{6} Ra (1 - Le) y^3, \tag{29a}$$

$$\theta = \frac{K_6}{K_1} x + K_8 + \frac{K_6}{2} y^2 + K_7 y, \tag{30a}$$

$$C = \frac{K_6}{K_1 N} x + K_{13} + \frac{K_6 Le}{2N} y^2 + K_{11} y. \tag{31a}$$

For the second case, Eq. (28) is still effective, but Eqs. (29a) ~ (31a) have to be changed to

$$p = K_3 - K_1 x - [Ru - Ra(K_8 - NK_{13})] y + Ra \frac{K_7 - K_{11} N}{2} y^2, \tag{29b}$$

$$\theta = \frac{K_9}{K_1} e^{K_1 x} + \frac{K_6}{K_1} x + \frac{K_6}{2} y^2 + K_7 y + K_8, \tag{30b}$$

$$C = \frac{K_9}{K_1 N} e^{K_1 x} + \frac{K_6}{K_1 N} x + \frac{K_6}{2N} y^2 + K_{11} y + K_{13}. \tag{31b}$$

The relation between abovementioned two solutions is similar to the relation between the solutions in the previous section. The solutions in this section represent the natural convection in porous media between two infinite horizontal parallel solid plates moving with the same velocity, and the fluid in the porous media is moving with the same velocity as the plates. However, the pressure, temperature and concentration of these solutions are functions of both x and y, so it is more valuable to be benchmark solutions.

3 The third algebraically explicit analytical solution

Similar to the preceding section, but choosing $v = K_4$ and $u = 0$ instead of $u = K_1$ and $v = 0$, with analogous derivation, the following solution is derived:

$$\begin{aligned} p = K_3 - (K_4 + Ru - K_5 Ra + K_{12} NRa) y \\ + Ra \left(\frac{K_8}{K_4^2} e^{K_4 y} + \frac{K_6}{2K_4} y^2 \right) \\ - NRa \left[\frac{K_{11}}{(K_4 Le)^2} e^{K_4 Ley} + \frac{K_{10}}{2K_4 Le} y^2 \right], \end{aligned} \tag{32}$$

$$\theta = \frac{K_6}{2} x^2 + K_7 x + K_5 + \frac{K_8}{K_4} e^{K_4 y} + \frac{K_6}{K_4} y, \tag{33}$$

$$\begin{aligned} C = \frac{K_{10}}{2} x^2 + K_9 x + K_{12} \\ + \frac{K_{11}}{K_4 Le} e^{K_4 Ley} + \frac{K_{10}}{K_4 Le} y. \end{aligned} \tag{34}$$

The physical feature of this solution is the natural convection in porous media between two infinite vertical parallel solid plates moving with the same velocity, and the fluid in the media is moving with the same velocity as the plates.

4 Concluding remark

Applying the approach of separating variables with addition, we derive three sets of simple and clear algebraically explicit analytical equations describing the natural convection in porous media with both temperature and concentration gradients. Besides the theoretical meaning, they are more suitable to be the benchmark solutions to check and improve the computational fluid dynamics and computational heat and mass transfer. Since the derived expressions are very simple, various typical curve drawings are omitted to save space.

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